A Surprising Constant Sum When a Figure Is Translated with Respect to Another Figure

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The problem discussed in this note was recently brought to our attention by Daniel Scher, who is "addicted to triangle shearing problems."[1] It was inspired by James Tanton and Brad Ballinger. The lovely and attractive phenomenon contained therein immediately caught our attention.

The Case of Triangles: A Geometric (Synthetic) Proof

Problem. An arbitrary triangle DEF sits inside a larger arbitrary triangle ABC. Show that the sum of the areas of the three green triangles of Figure 1 is unchanged as DEF is translated within the larger one.[2]

Proof. Daniel Scher gives a six-step proof using shearing. We follow his idea with two movements in the second step:

1. Construct a parallel to DE through A, a parallel to EF through B, and a parallel to FD through C, yielding PQR (Figure 2a).
2. Move the vertices A, B (as vertices of the green triangles) to position Q, not affecting the areas of the triangles, that is, $|AED| = |QED|$ and $|BFE| = |QFE|$ (where $|\cdot|$ means "area" (see Figure 2b).

Now we look at quadrilateral $DQFC$ in Figure 2b. On the one hand, its area can be calculated easily by $\frac{1}{2} \cdot DF \cdot \left(h_1 + h_2\right) = \frac{1}{2} \cdot DF \cdot d(Q, PR)$[3], which does not depend on the initial position of DEF and is therefore constant. On the other hand, it consists of the four triangles DEF, CDF, QFE, and QED, from which the area $|DEF|$ is also constant (the initial smaller triangle), and $|AED| + |BFE| = |QED| + |QFE|$. Hence, the sum of the areas of the three green triangles is also constant! Exactly in the manner described, this works for translations of the smaller triangle DEF not only in the interior of ABC but even in the whole of PQR. In other words, if DEF lies, for example, in CBR, nothing changes.

Remark. When using directed (oriented) areas and directed (oriented) distances (for h_1 and h_2), the property of the constant sum of the areas of the green triangles holds when DEF is moved anywhere in the plane.

Note that the above proof never made use of the fact that DEF is smaller than ABC. This is indeed not necessary. To speak of a smaller triangle sitting in a bigger one and being translated is a good formulation for becoming familiar with the situation because it is easier to imagine, but it is irrelevant.

An Analytic Proof for Polygons and Further Generalizations

Let us formulate the findings from above concerning triangles, more generally, namely for arbitrary polygons, as Theorem 1.

Theorem 1. *Let $P = P_1 \ldots P_n$ and $Q = Q_1 \ldots Q_n$ be two arbitrary n-gons in the plane. Then, using directed (oriented) areas, the area sum of the triangles $P_iP_{i+1}Q_i$, namely $\sum_{i=1}^{n} |P_iP_{i+1}Q_i|$, where the sum is cyclic, is independent of translations of both n-gons $P_1 \ldots P_n$ and $Q_1 \ldots Q_n$ (Figure 3a with $n = 5$).*

In an earlier version of this short note, we had further geometric proofs (quadrilaterals, polygons, and tetrahedra in three dimensions). Then Sergei Tabachnikov, editor of the *Mathematical Intelligencer*, remarked that all these cases can easily be proved analytically (many thanks!). In the following we mention his ingenious short proof.

Proof. The area sum of the theorem is given by $S = \frac{1}{2} \sum_{i=1}^{n} \det(P_i - Q_i, P_{i+1} - Q_i)$, where the sum is cyclic. We have

$$\det(P_i - Q_i, P_{i+1} - Q_i) = \det\left(P_i, P_{i+1}\right) + \det(P_{i+1} - P_i, Q_i).$$

[1]See Daniel Scher. A Beautiful Application of Shearing. Available at https://www.sineofthetimes.org/a-beautiful-application-of-shearing/ (2025). Accessed November 28, 2025.

[2]Before continuing further, the reader may wish to first explore the problem dynamically at https://dynamicmathematicslearning.com/tanton-ballinger-area-sum-result.html.

[3]By $d(Q, PR)$ we denote the distance of Q to line PR.

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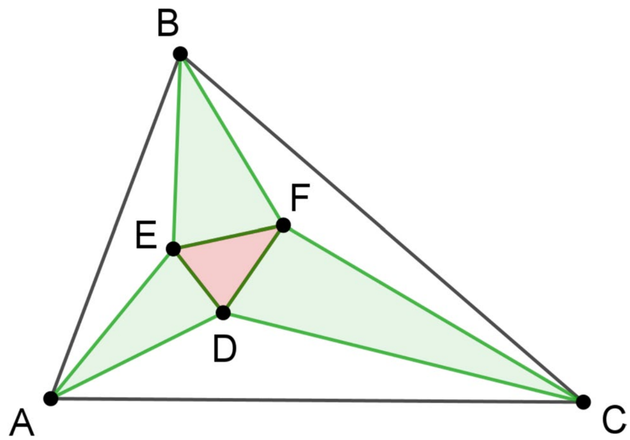

Figure 1. A small triangle inside a larger one.

Translating the polygon Q replaces Q_i by $Q_i + V$, where V is a fixed vector. Then $\sum_{i=1}^{n} \det(P_i, P_{i+1})$ is not affected, and $\sum_{i=1}^{n} \det(P_{i+1} - P_i, Q_i)$ is changed by

$$\sum_{i=1}^{n} \det\left(P_{i+1} - P_i, V\right) = \det\left(\sum_{i=1}^{n} (P_{i+1} - P_i, V)\right) = 0,$$

since every vertex of P appears there twice with opposite signs. Hence S is also unchanged. This argument works for every $n \geq 3$, showing impressively the power of determinants, including the corresponding properties and rules.

It even works, mutatis mutandis, in higher dimensions, e.g., in dimension 3 with tetrahedra instead of triangles (Figure 3b): The oriented volume of the four green tetrahedra in Figure 3b is independent of translations of both tetrahedra $ABCD$ and $EFGH$.

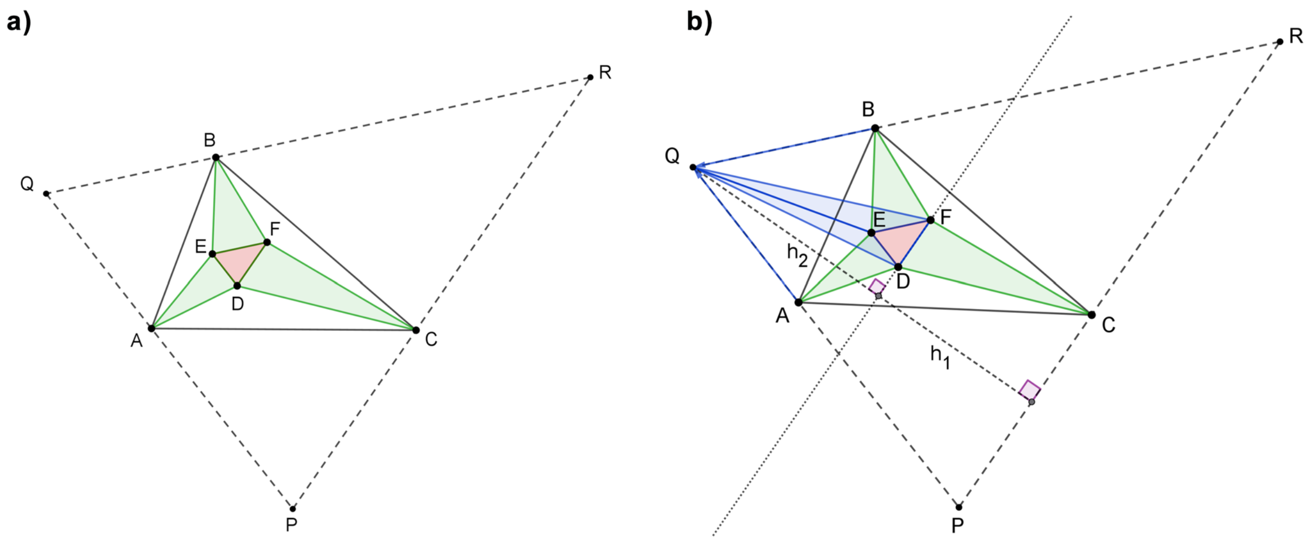

Figure 2. (a) The parallels to DE, EF, FD establish a bigger triangle PQR. (b) The sum $h_1 + h_2$ is constant, so the area of $DQFC$ is constant.

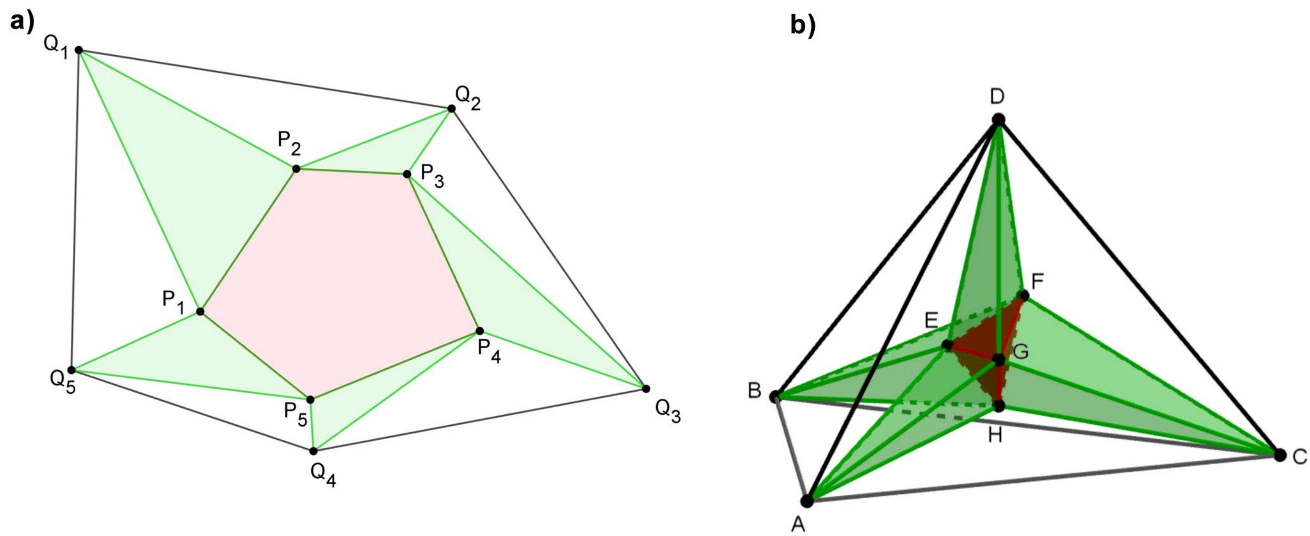

Figure 3. (a) Two pentagons. (b) Three-dimensional analogy with tetrahedra.

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