Arithmetic operations in a Physical context.

As an example, consider how we do many physical actions which are then abstracted from their physical context by formal arithmetic operations. Sometimes we condense a rich variety of activitics into a single formal algorithm. Think about how you would:

- *Share 25 buttons equally among 4 people.*
- *Find how many shirts with 4 buttons each can be made with 25 buttons.*
- *Put 25 people into taxi cabs holding at most 4 people each.*
- *Make shirts from 25 yards of material needing 4 yards each.*
- *Share 25 yards of material among 4 people.*

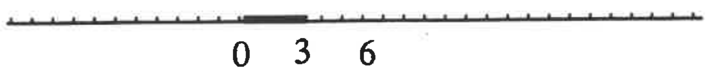
Make a picture to indicate how you would carry out these questions. You will see that they require different physical actions. This rich variety of physcial activities is all condensed into the formal arithmetic operation of dividing 25 by 4. This is both the power of abstraction, as well as the pitfall. Also, the remainders mean different things in the different contexts. The distinctions of .25, 1/4 and r=1 do not fully convey these differences.

Numbers and Lines

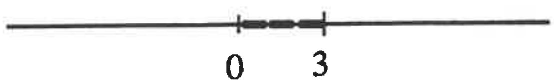
A. There are several ways to think about what the number 3 on the number line might mean.

1. The name of a point

2. The length of the segment between 0 and 3

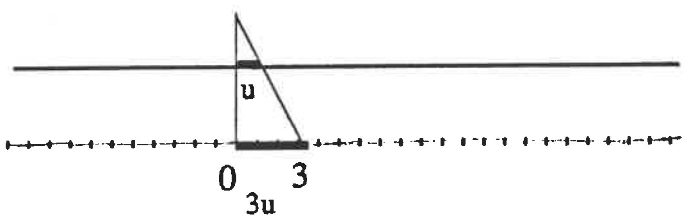


3. A translation to the right of 0 by u repeated 3 times.

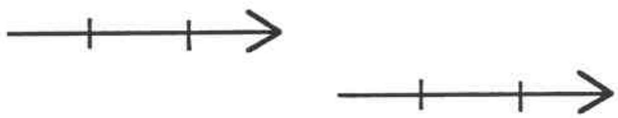


u = ▬ $3 = T_u(T_u(T_u(0)))$

4. A scalar magnification by 3 of the unit u.



5. Any directed segment with length 3 and direction right.



It is important to note that in different contexts we will be using different meanings for the numbers on the number line: sometimes the name of points, sometimes lengths, sometimes vectors.

B. ***Numbers on the line have a history: they may be constructed from different actions.***

Given a segment or position, how do we assign a number?
Given a number, how do we assign a position or segment?

A good place to start is to assign unit length u to some segment.[1] The procedure for placing number coordinates on a number line intrinsically determines natural operations on these numbers. Starting from a 0 point and a unit **u**, we can build up the coordinate numbers on the line by translating or by dilating:

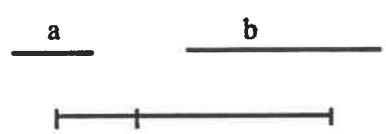
Repeatedly laying this segment end to end, we can get all segments whose length's are whole number multiples of u. Intrinsic to this process of building whole number length segments is the geometrical operation of **translation.**

It is important here to think of the numbers not only as static entities, but as processes: each number carries its history with it. For example, if we think of 2 as the position on the line gotten by translating by two units from 0, then we have:

$$2 = T_u(T_u(0)))$$

Since these numbers are built by translating, a natural operation between these LN's is to translate one by the other: this is the same as combining by composition the two processes used to get the numbers from the unit u. If we look at the results, this operation corresponds in the arithmetic world to addition. $a+b = T_b(a)$

To add two segments a and b, simply place them end to end.



This is the same as translating one by the other:

You can easily show that $T_a(b) = T_b(a)$ (= a point or coordinate!)

---

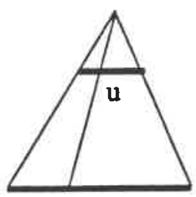
1 Note that we are assuming Euclidean space here, and particularly that there is no special segment we should call 1, and that the space is uniform.

C. Linear Numbers as Dilations and rotations of u.

We can construct the numbers or coordinates by dilating or magnifying or scaling the unit.

In order to scale we need a unit.
We need to magnify in order to find rational segments or coordinates without re-scaling.
We need to go off of the line to magnify or dilate.

Think about what scaling or magnification means. How does $M_3(\mathbf{u})$ differ from repeated addition of u? In what sense is $M_3(\mathbf{u})$ the same or different than u+u+u? The end position is the same, but the process is very different.

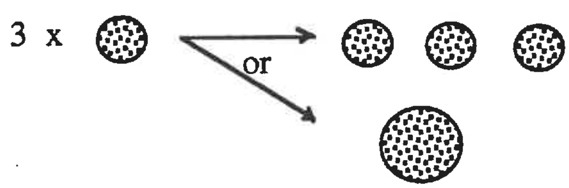


3u

3u as scaling u by 3

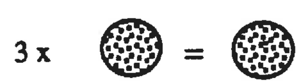
3u as repeated translation

Notice that scaling by 3 is not necessarily the same as repeatedly adding something to itself three times.
For example, think about the difference between 3 cookies, and one large cookie three times as big as an original. These are very different processes.



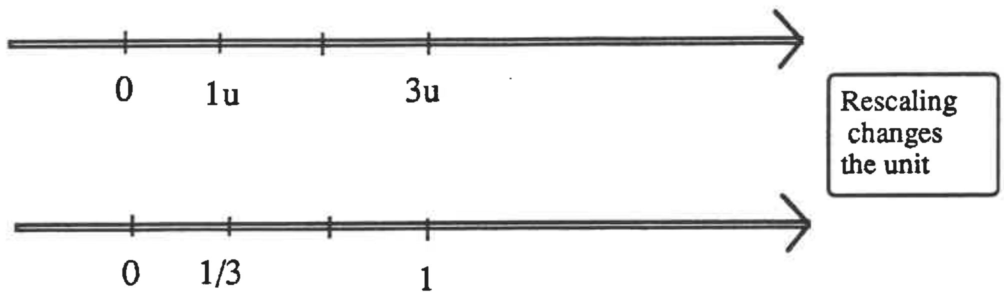
The analogy for repeated addition is to have three chocolate chip cookies. The analogy for magnifying, is to have a chocolate chip cookie three times as big. Notice that this last phrase "three times as big" is also ambiguous unless we define our term more carefully. Three times as big may mean three times the radius, or the volume!

One student, asked to draw a picture of a cookie three times as big drew the same cookie: "Its three times as thick" she said.



D. In order to place the rational numbers we have to go off of the line.

How do we determine the segments whose lengths will be fraction numbers? Translating the unit will not get us the coordinates on the line which are between whole numbers. If we already have a number line with the whole number multiples of u, we can rescale to find 1/3. But rescaling means changing our unit. We will find no new coordinate points in this way.



In order to locate rational numbers from the unit without rescaling we can dilate or scale the unit[2]. How do we scale or magnify a segment? If we want to magnify a unit by 3, for example, we need to know the meaning of the scaling factor 3 in the geometric sense.

a. We first need to have two segments which we know are in a 3:1 ratio.
b. One test of this ratio is that 3 copies of the smaller fit onto the larger.
c. In this sense we do need to have established whole number multiples of some segment by a repeated addition process.

So we shall assume we have segments representing all whole multiples of our unit. To get fractions without re-scaling it is natural to use the properties of similar triangles. Suppose,that we wish to construct the segment corresponding to m/n for any whole m and n. Here are two procedures.

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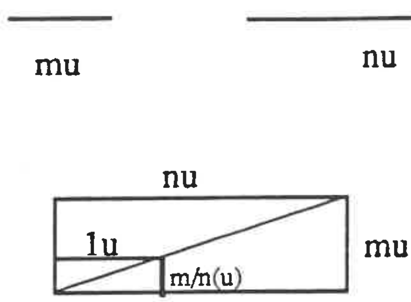
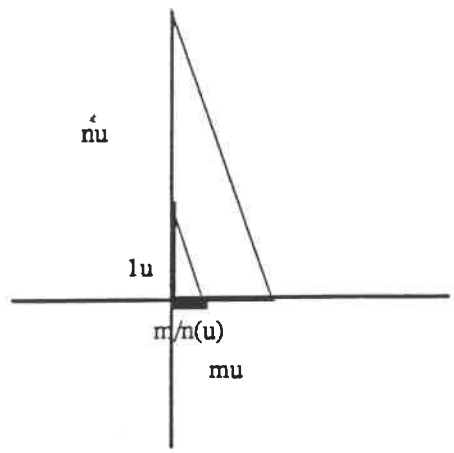
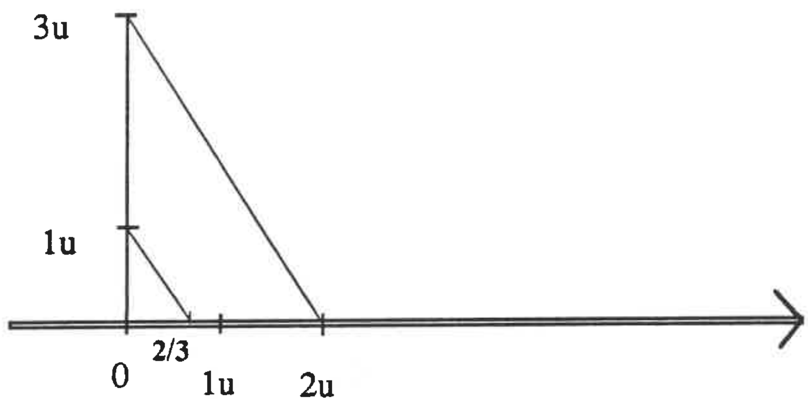
2 We could also successively approximate the position of 1/2, for example, by picking a point between 0 and 1 and measuring which was closer. Then place a point symmetrically near the other end and again pick a point. etc.





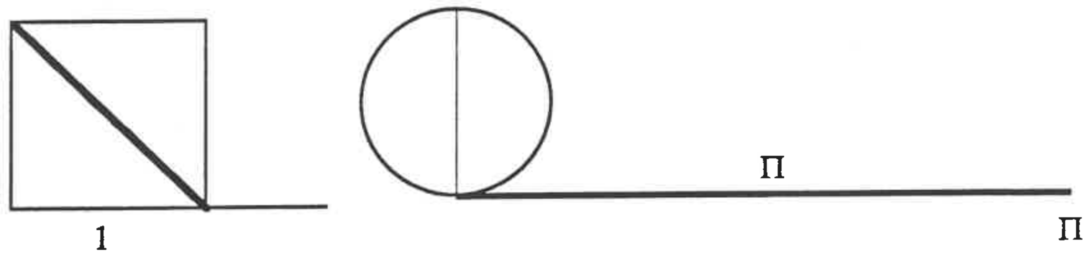
Thus we can construct segments of any rational length. The similar triangle procedure is essentially the operation of **scaling**, which is thus built into the construction of the rational number segments.



We now have two kinds of numbers to work with. We have lengths of segments expressed in numbers of units. We also have ratios of segments, the scalar numbers multiplying the unit--represented as triangles.

"Irrational" numbers.

In order to construct segments of irrational length, ie. those which are not commensurable with the rational length segments, we need to use other procedures--which are finite for only a small class of special numbers such as $\sqrt{2}$ and π.



Diagonal $=\sqrt{2}$ Roll out unit circle to get segment Π

In general, in order to place the irrational numbers, we will have to use a limit procedure. However, if we assume we already have all size segments at our disposal, we can lay out these segments in size order by comparison to u. In this case, there is a different meaning to "ratio". Evey segment A will form a rectangle with u, and the ratio of A to u is measured by the angle of the diagonal. See the discussion later on in the section on different meanings of multiplication.

Numbers to the left of 0.

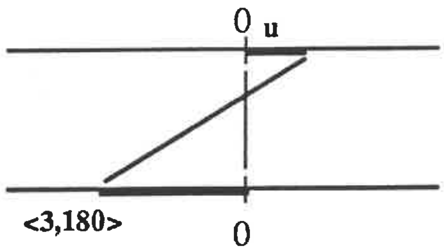
Numbers on the line have an intrinsic direction. When we call the numbers to the right of 0 *positive* numbers, we mean geometrically a direction along the ray that we can call 0 degrees.

TRANSLATION: We can build a number to the left of 0 as a translation followed by a reflection about 0 of 0. The operation of subtraction means rreflecting and then translating. This difference in the order of operations helps to distinguish subtraction from the construction of the opposite numbers.

As an example, we can now make sense of a problem like 5-(-8). To subtract is to reflect the number and translate. But -8 is already a reflection about 0. So -(-8) means translate 0 by 8u, reflect, reflect. The result is the same as translating only, so 5-(-8) is the same as 5 translated by 8 or 5+8.

DILATION: "Positive" numbers are found by dilation: 3 means D_3(**u**). We will use the notation <3,0> for the number +3. This is more cumbersome notation than +3, but will make much clearer the history of the number: 3 is a dilation by 3 and rotation by 0 of **u**.

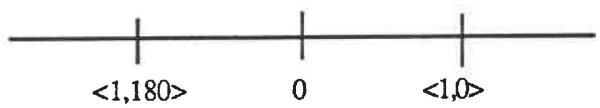
To ***construct*** the numbers to the left of 0 by dilation--whether we think of these as names of points or directed segments--rotate by 180 degrees. Thus, the number <3, 180> is the unit segment **u** dilated by 3 and flipped.

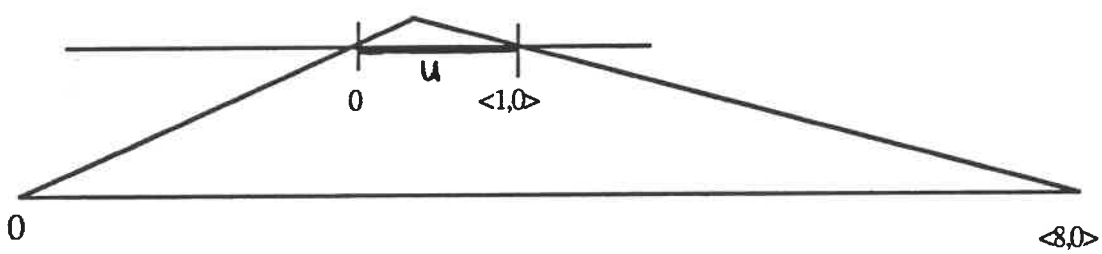


<3,180> is the way of writing a dilation of u by 3 and rotation by 180.

[Note: Gauss, in the 19th century, tried to come up with a better terminology than "negative". He wanted to call the positive numbers "direct" and the opposites "indirect", while those to the side he wanted to call "lateral numbers".

So, intrinsic to the construction of the numbers as scalar multiples of a unit is the dilation or scaling transformation. <8,0> means the unit segment dilated by 8, or the position on the number line that 1 is mapped to by dilating the coordinate system by 8. <8,180> means a dilation and rotation (convince yourself that these operations commute, whereas translation and rotation do not.)





NOTE: we should emphasize here that when we think of numbers as actions, they operate on the whole line. For example, when working with inequalities, 3<5 means that 3 is to the left of 5. What operations on the whole line will preserve the order of the line and which will not? Of the basic operations, only multiplying or dividing by -1 will reverse the order--because it switches the direction.

D. Intrinsic to the construction of numbers is the basis of the operations which can be used to relate the numbers.

When we want to perform operations on or between the segments, we can translate the geometric operations into arithmetic by thinking about the operations that are intrinsic to the way we built the numbers. Each number comes with its own history. The numbers we assign to geometric objects are not only static objects, they also stand for processes, and actions.

For example, if we build 3u from u by repeated translation, then 3u means a translation of u end to end three times. To combine 2u and 3u we compose these actions to get 5u-- a translation of u 5 times end to end, or placing 3u and 2u end to end. Thus we correlate the geometric operation of translation with the arithmetic operation of addition.

For the segments like 1/2(**u**) and 2/3(**u**), the operation built into the numbers is scaling--not translation. So it is much more natural to compose or combing fraction segments by multiplying then by adding. (1/2)(2/3) means a composition of a 1/2 scale and a 2/3 scale. This can be done with similar triangles, and is easily shown to be equivalent to a 1/3 scale, giving us the answer 1/3(u).

At this point it is important for the reader to think about the following:

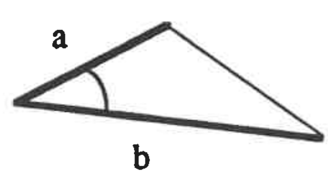
Given two segments a and b, what do we mean by the product of a and b (in other words, what does the arithmetic operation of multiplication mean in the geometry world.)

Multiplication is more complicated than addition. There are several meanings for the multiplication of LN's geometrically, even though the numerical value of a product is formally easy to find for whole numbers.

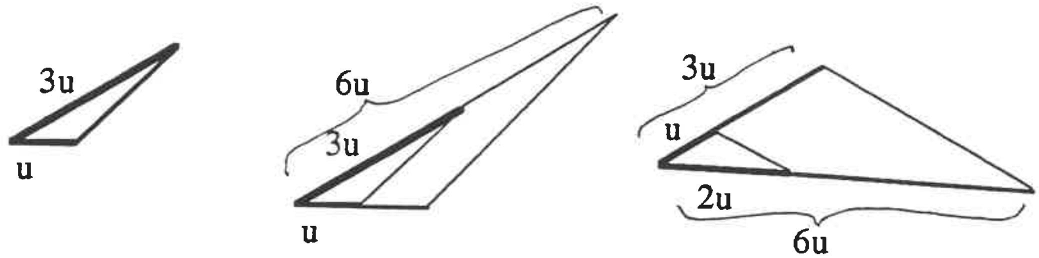
a. Multiplication as repeated addition: 3x2 means 2 taken three times. Geometrically, 3b as repeated adding means put segment b next to itself 3 times. To multiply b by 3 we translate (compose the translation) 3 times:

b. Multiplication as scalar multiplication: 3X2 means the LN 2u magnified 3 times. Here, the **2** is not the same kind of number as 3. The second item may be a geometric figure. *The natural way to scale is to use similar triangles.* We have to think of the number 3 as a ratio 3u/1u and apply it (the triangle) to 2.

Given two segments a and b, a triangle with sides a and b gives us a model for ratio.



Let us call a triangle such as constructed above with sides **a** and **b** a triangle with ratio **a/b.** Any other triangle similar to this one will also have the ratio **a/b**. If we have a unit length **u**, we can "apply" a triangle to **u** to construct a length **a/b**(u) in two ways.



We need a unit to multiply, but we DO NOT need a unit to scale a segment **c** by a scalar **a/b**. Note the similarity of this operation to the method we used to construct the "rational" segments.

This distinction between the two terms in the multiplication helps us to see why the natural order operations is multiplication before addition.

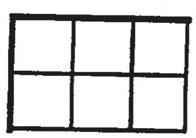
The expression **2** + 3x**5** means **2u** + 3x**5u.**

2 means 2u, **5** means 5u, but 3 means a scalar.

Therefore, 3x**5u** is one whole term.

An important question here, is "What does it mean to multiply (as scaling) by an irrational number?" You should see that, again, it depends on the definition/history of "irrational".

c. Another geometric meaning of multiplication of A x B is the area of a rectangle with sides A and B. Notice that we are trying to define operations on LN's, but the result is not an LN. Area is in terms of square units. Also certain operations will work on the area AB which will not work on the pure number AxB without some care. For example, square roots of area make sense, but square roots of lengths do not, as we pointed out earlier.



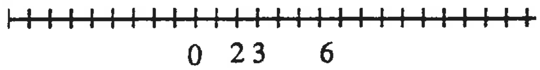
2" x 3" = 6 sq in.

But how do we get linear units?

Three observations about multiplying as Area.
-No unit is needed to represent the product: the rectangle as a region is invariant under change of units.
-We may multiply any two segments in this way, regardless of whether they are commensurable or not (ie irrational).
-Multiplying increases dimension in this model. Powers more than 3 are hard to visualize, as are fractional dimensions.

Two problems with the number line is that there is an implied direction, and there is length or "spread" between numbers. The latter problem is related to the need for a unit.

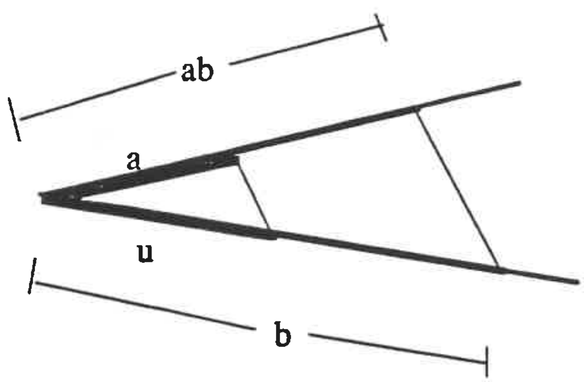
i. Consider 2 x 3 = 6. If we correlate 2 with a point on the number line as 2", and 3 as 3", then we ask for the point on the number line which represents 2" x 3" we are stuck: because the meaning of the x is not clearly defined. Should the answer be 6 square inches? If so, how can we put this on an inch-unit number line?



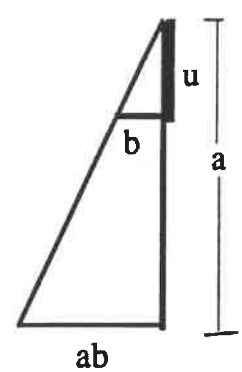
ii. Consider the geometric process: "given a region with a certain area, find the side of a square whose area is equal to the given region." This is a well-defined process for many regions. We model this process within the arithmetic world by square-roots, and we write / 4 = 2 without worrying about the original context or units. However, when we translate back to the geometry world, and 4 is a number on the number line representing a length, we cannot pur the /4" on a number line--this item would be 2 but have dimension 1/2!

d. Multiplication as area does not satisfy the condition that we require the product to be the same kind of number as the original. In order to turn the geometric multiplication as area into a closed operation on LN, we have to show how to translate the quantity of area AB into an LN. This demands a suitable unit U, and the length of AB will depend upon the unit U.

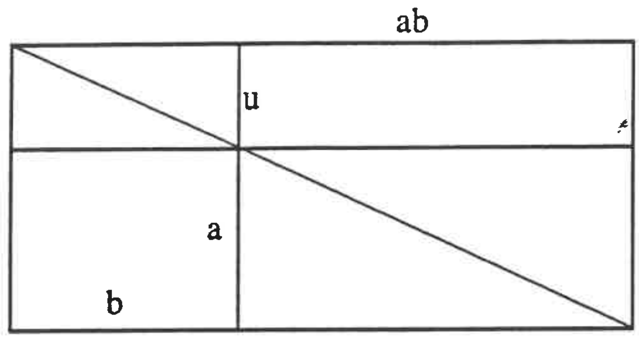
There are at least three good pictures for finding the segment ab--both using similar triangles. Convince yourself that these pictures work:



or



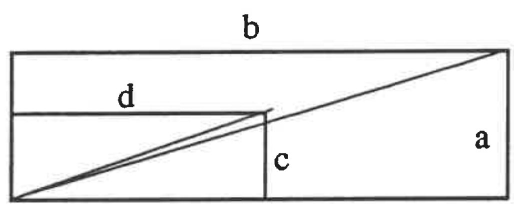
Here is another way to do it, with rectangles and similar triangles:



Construct rectangle a by b.
Then extend side a by length u.
Complete the rectangle u by b.
Extend the bottom line b.
Draw the diagonal from the upper left corner of rectangle u by b. Complete the picture to get length ab.

ASIDE:

If we do not FIRST import the rational numbers from the arithmetic world, then we can simply construct ALL possible segments from ratios of existing segments APPLIED to the unit segment we have chosen. We do not even need to know if the segments **a** and **b** are commensurable. We can order the ratios easily by a diagonal test:



We see that $a/b < c/d$ because the diagonal of rectangle ab makes a smaller angle with the horizontal than cd.

E. Multiplication of plane numbers.

The natural operation for numbers constructed by dilation and rotation is a composition of dilations and rotations.

For example:

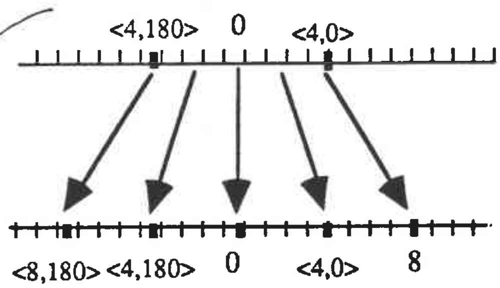
<8,180> = $R_{180} \circ D_8(\mathbf{u})$ and <1/2,180> = $R_{180} \circ D_{1/2}(\mathbf{u})$

The natural operation between <8,180> and <1/2,180> is to compose rotations and dilations. The result is <4, 360> = <4,0>.

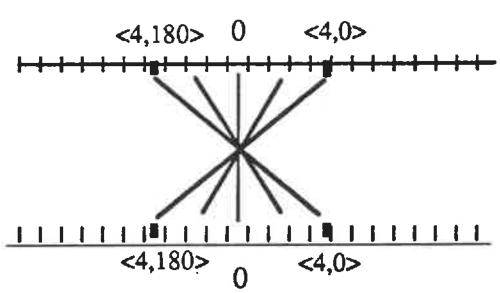
Don't think of 2 as a static item, but as dynamic: a verb or operation.

We should also think of the operation of multiplying by <2,0> or by <2,180> as an operation on the whole number line.

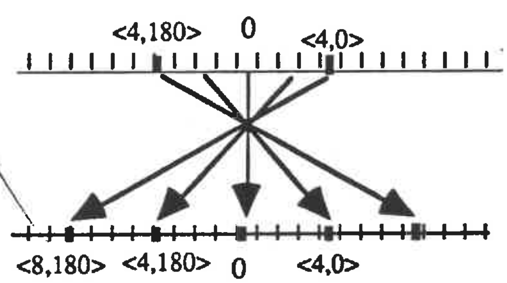
Sorry for skewed diagrams: Printer won't print as on screen!?



Multiplying by <2,0> is the same as scaling by 2 around the origin.



Multiplying by <1,180> is the same as rotating 180 degrees around 0.



Multiplying by <2,180> is the same as scaling by 2 around the origin and rotating 180.

This operation of rotation and dilation is easily seen to correspond to the arithmetic operation of multiplication.

F. Multiplying by - as rotation by 180.

We now have a meaning for the arithmetic operation of multiplyig by a "negative" number. The order of operations is different for 2 x -3 and for -2 x 3, but the result is the same:

$2 \times -3 = \langle 2,0 \rangle \circ \langle 3,180 \rangle = R_0(D_2(R_{180}(D_3(\mathbf{u}))))= R_{180}(D_6(\mathbf{u}) = \langle 6,180 \rangle = -6$

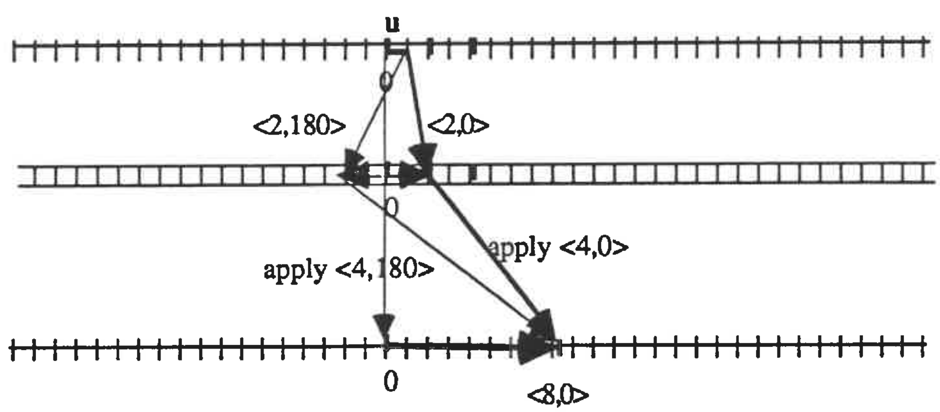
$-2 \times 3 = \langle 2,180 \rangle \circ \langle 3,0 \rangle = R_{180}(D_2(R_0(D_3(\mathbf{u}))))= R_{180}(D_6(\mathbf{u}) = \langle 6,180 \rangle = -6$

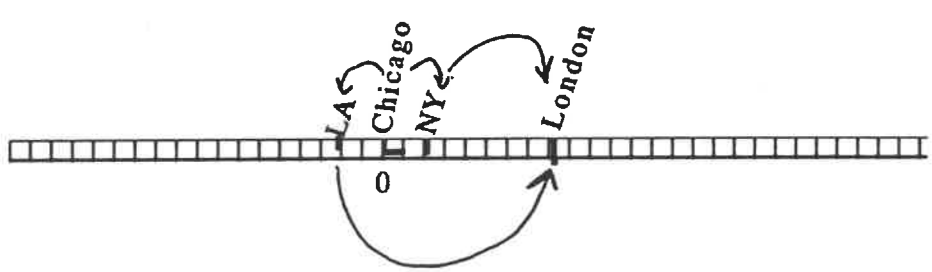
For perhaps the first time, we see a natural reason why the product of two "negative" numbers is positive--intrinsic to the meaning of negative number.

$-2 \times -3 = \langle 2,180 \rangle \circ \langle 3,180 \rangle = R_{180}(D_2(R_{180}(D_3(\mathbf{u}))))= R_{360}(D_3(\mathbf{u}) = \langle 6,0 \rangle = 6$

Students should check and **see** all these operations on line diagrams such as the ones above. For more detail, start with **u** and perform the operations.

Notice that although <2,0> o <4,0> gives the same result as <2,180> o <4,180> the processes are very different. The first might be like starting at Chicago and flying to NY and then to London. The second might be like starting at Chicago, flying to LA, and then to London. This is a particularly important point when we look at the whole plane.



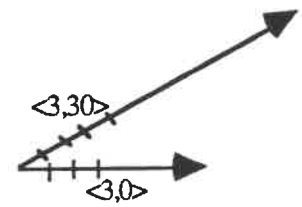


G. Plane Numbers

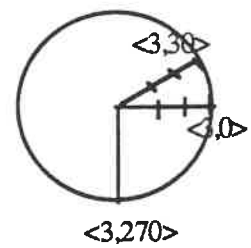
The plane numbers are built up from the Unit u by rotation and dilation. The natural operation is a composition of these operations. This will also be a natural extension of the arithmetic operation cf multiplication.

There is no intrinsic reason we must restrict ourselves to numbers in the particular direction we have called 0 (or 180). We can take a ray in any other direction, such as 30 degrees, and find a whole ray full of numbers.

For every angle, such as 30, we have a whole number ray.



For every positive number such as 3, there are an indefinite number of numbers 3 units from our origin.



A model for this kind of number system is to think of directions. Perhaps a better notation for angles would be something like 1/12 turn instead of 30.

Plane Numbers as Solutions to Equations.

In order to provide another motivation for moving off of the line, we will look at the solutions to some simple equations in our new notation. [NOTE: This sequence is inspired by David Bock.]

The solution of linear equations suggests the consideration of negative numbers--for example in trying to solve a simple equation such as $X + 9 = 4$. It is easy to solve subtraction problems in the translation construction of numbers, but it is neither obvious nor easy to add and subtract numbers in rotation/dilation notation. The +,- number system is made for addition and subtraction. Historically, mathematicians did not want to consider the use of negative numbers until the solutions of cubics made the use of plane numbers realistic, and only then was the use of negative numbers (as rotations) accepted as well.

Consider the second degree equation $x^2=16$. In our new notation, we are asking for < , > such that :

$$< \, , \, > \circ < \, , \, > = <16,0>$$

The rotation and dilation model is natural for multiplying. We find:

<4,0> o <4,0> = <16,0>.
but also <4,180> o <4,180> = <16,360> = <16,0>

So we have two solutions to our problem. Note that in order to emphasize that <4,0> means a process of dilating the unit, we put in the unit segment also as our starting position. ie:

Start with **u**, dilate by 4 two times.
Start with **u**, dilate by 4 and rotate by 180 two times.

Now continue. There are two solutions to

< , > o < , > = <4,0>

<2,0> o <2,0> = <4,0>.
<2,180> o <2,180> = <4,360> = <4,0>

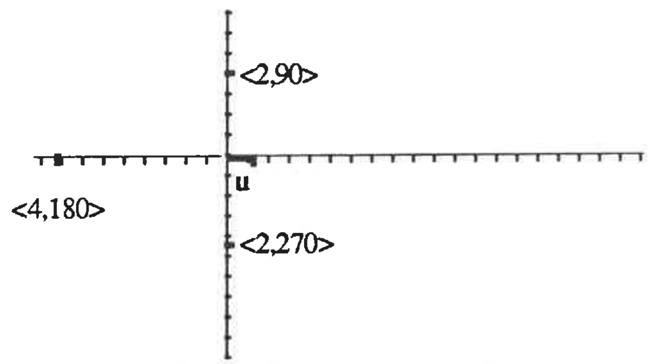
So far we have been able to satisfy our equations with numbers from the line.

We usually think of the number line as "all" the numbers. But the number line is not a rich enough environment to hold the answer to these problems. Here we have a whole plane full of numbers.

To see this, think about **< , > o < , > = <4,180>** ?

Students will not usually take a long time to see that:

u ---> <2,90> ---> <2,90> o <2,90> = <4,180>.
and also
u ---> <2,270> ---> <2,270> o <2,270> = <4,540> = <4,180>

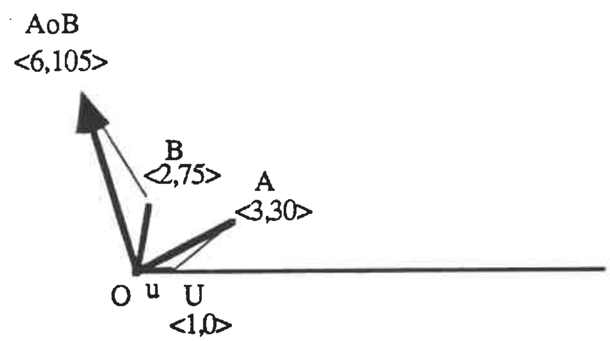


Now keep finding square roots of <2,90> and so on. There are a potentially infinite number of routes from Chicago to London.

This approach gives us a natural introduction to numbers filling up the whole plane. We can continue to explore the geometric meaning of plane number concepts, and then show that these geometric operations are equivalent to operations on complex numbers as we usually define them.

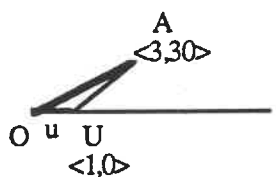
H. Multiplying Plane Numbers as composition.

To return to our sequence of ideas, each plane number has length and direction. Having constructed plane numbers by dilation and rotation, and seen a motivation from solving equations, we can now define our operations between plane numbers intrinsic to their construction in this way.

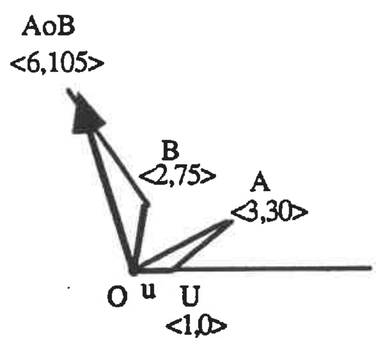


We multiply plane numbers using the transformations built into their construction. <3,30> x <2,75> = <6,105>. This product is found by dilating by 2 and 3 and rotating by 30 and 75. Practically, it is found by "applying the triangle AOU to OB.

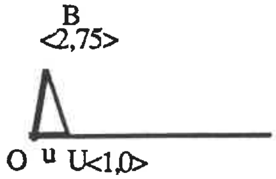
Let us look a little more carefully at our operation and see how it grew out or our construction of the numbers in the first place.



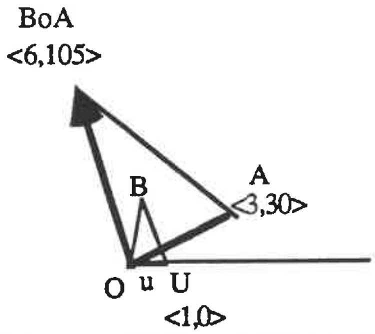
Plane number A has intrinsic to it a whole triangle. The important features of the triangle are the ratio A/u =3 and angle AOU = 30. This triangle represents A as an ACTION, and we can apply this triangle to B as an operation--by keeping the ratio and angle the same.



B is another plane number. When we operate on B using A, we apply to B a triangle similar to UOA: matching up UO to BO. The result is directed arrow AoB, whose length 6 is the composite of the dilations 2 and 3, and whose angle is the composite (sum) of 30 and 75.



Plane number B has intrinsic to it a whole triangle. The important features of the triangle are the ratio B/u =3 and angle BOU = 75.



When we operate on A using B, we apply to A a triangle similar to UOB: matching up UO to AO. The result is directed arrow BoA, whose length 6 is the composite of the dilations 3 and 2, and whose angle is the sum of 75 and 30.

We emphasize here that an origin and a UNIT **u** is needed to multiply. We need not have the unit horizontal, but in order to multiply by plane number A we need to use A/**u** and to use the direction of A based on **u**. What we are saying, is that mulplication is really a proportion. A:u = AB:B OR B:U = AB:A. Three of the terms in the proportion are known, and the other (AB) is to be found. *This has direct consequences for our linear operations as well!*

Exercises:

a. For each plane number A we can define its length | A | , the length of the vector, as the distance to the origin. In the dilation/rotation notation for plane numbers the length is always positive.

b. Draw all plane numbers with length 1. Draw all plane numbers with direction 60. What happens when you multiply any plane number by a plane number with length 1? By a plane number with direction 0, or 180 or 90? What is the effect as a geometrical transformation?

c. Find ways to get <1,180> by squaring a plane number. Draw pictures. How many ways can you do it?

d. Draw pictures for powers of <1,90> such as 2,3,4 ...100. What patterns do you notice?

e. Play with multiplying numbers z which are on, in, and outside of the unit circle: what effect does multiplying by z have in the different cases? Think about numbers on the unit circle to start with.

f. Show that translation works as the geometric equivalent of adding for plane numbers. show that this is the same as "vector" addition--putting two vectors beginning to end.

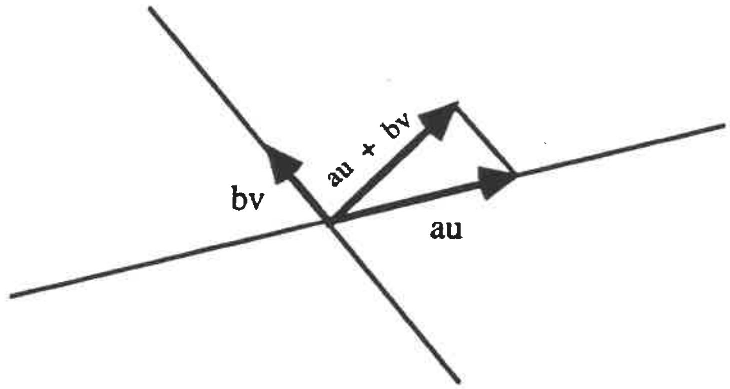
g. Draw pictures to show how to divide a plane number b by a. Think about the four terms of the proportion with a, u, b and b/a.

h. Check that all rotations and translations preserve distance and angles, and thus preserve congruence.

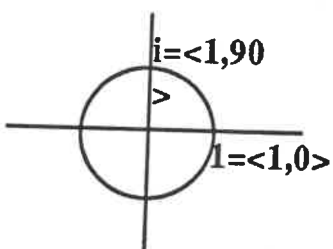
Check/show that magnification/dilation/scaling preserves the angles of a given figure: that is shape but not size. Which of T, D, R commute with each other?

I. Plane Numbers built up from Addition/Translation

We may also build the plane numbers up by addition/translation. Consider all the numbers on two number lines which intersect at their common 0 point. Let the lines have units u and v, where u and v are directed unit segments on the lines. We think of the numbers on the lines as scalar multiples of u and v. Plane numbers can be considered as the set of all vector sums au + bv .



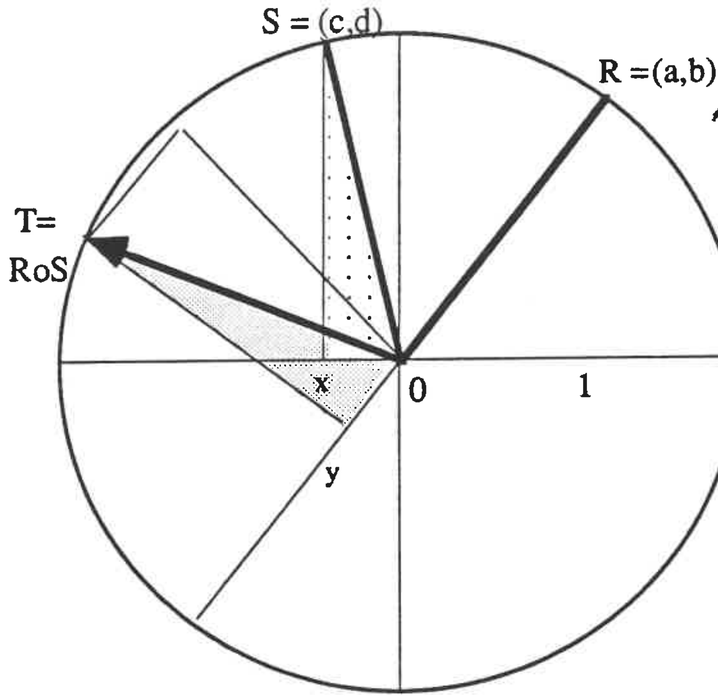
Using this construction of the plane numbers it is easy to add numbers, but harder to multiply. If we use the standard notation that one of the axes is the linear (Real) numbers with unit **1**, and the other axis is perpendicular to the original with unit designated as **i**, then we have the standard rectangular representation of "complex" numbers. We use notation (a,b) = a**1** + b**i** or just a+bi.



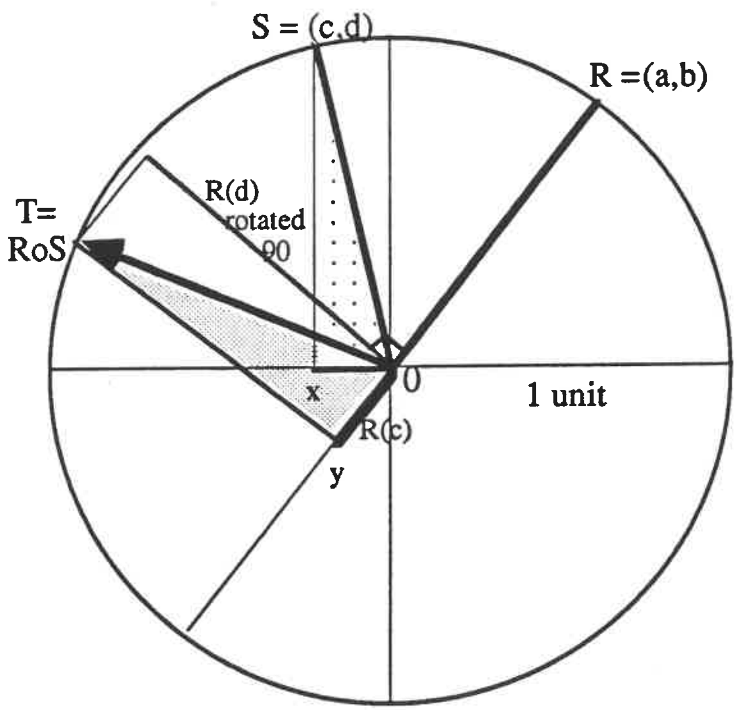
i=<1,90> is a plane number whose square is <1,180> ie -1

We now need to show the **equivalence** of multiplication as constructed from rotation and dilation and multiplication by the distributive law on (a+bi)(c+di).

a. Start from the MoR construction, and the corresponding meaning of multiplying plane numbers. It is easy to see that <p,r>o<q,s> is the same as pq(<1,r>o<1,s>). So we really only need to show that <1,r>o<1,s> will give the same result as multiplying the corresponding rectangular coordinate vectors [a,b] and [c,d] .



RoS is the product of the two vectors R and S on the unit circle--so it is a rotation of S by the angle of R. Therefore, by AAS, the shaded triangles are congruent. (triangle Ty0 and Sx0).



|0y| = |0x|=|c| . Thus the vector oy, in the opposite direction from R and length |c| is equal to cR. This is c(a+bi).

|Ty|= |Sx|= |d| . Thus the vector yT, in a direction 90o to R and length |d| is equal to d(1,90)oR.
Notice that (1,90)oR has coordinates (-b,a), which is -b+ai. So yT = d(1,90)oR = d(-b+ai)

Thus RoS is T, which is oy + yT as vector addition, which is c(a+bi) + d(-b+ai).
This is the same result one gets by multiplying (a+bi)(c+ci) by distributive law, and using i i = -1.

b. To show the equivalence in the other direction, draw pictures for the following series:

Start with vectors a+bi and c+di.

1. Show that multiplying by [a,0] magnifies but doesn't rotate b+ci.
2. Show that multiplying by i rotates by 90 but doesn't magnify.
3. Show that bi(c+di) rotates by 90 and magnifies by b
4. Now write (a+bi)z = az + biz [these are equal in the rectilinear sense.] See that az and biz must be at right angles to each other (by 1. and 3. az is in direction z, biz is 90^o to z). Also az is a magnification of a by $|z|$, and biz is a magnification of b by $|z|$. So the rectangle az by biz is similar to the rectangle a by bi. So the resultant az+biz, which is (a+bi)(c+di) is a magnification of (a+bi) by $|z|$ and a rotation by the angle of z.